

BSE-PROPERTY FOR SOME CERTAIN SEGAL AND BANACH ALGEBRAS

M. FOZOUNI AND M. NEMATI

ABSTRACT. For a commutative semi-simple Banach algebra A which is an ideal in its second dual we give a necessary and sufficient conditions for an essential abstract Segal algebra in A to be a BSE-algebra. We also show that a large class of abstract Segal algebras in the Fourier algebra $A(G)$ of a locally compact group G are BSE-algebra if and only if they have bounded weak approximate identities. We study the BSE-property of some certain Segal algebras which introduced recently by J. Inoue and S.-E. Takahasi which implemented by local functions. Finally we give a similar construction for the group algebra which implemented by a measurable and sub-multiplicative function.

1. INTRODUCTION AND PRELIMINARIES

Let G be a locally compact abelian group. A subspace \mathcal{S} of $L^1(G)$ is called a (Reiter) Segal algebra if it satisfies the following conditions:

- (1) \mathcal{S} is dense in $L^1(G)$.
- (2) \mathcal{S} is a Banach space under some norm $\|\cdot\|_{\mathcal{S}}$ such that $\|f\|_1 \leq \|f\|_{\mathcal{S}}$ for each $f \in \mathcal{S}$.
- (3) $L_y f$ is in \mathcal{S} and $\|f\|_{\mathcal{S}} = \|L_y f\|_{\mathcal{S}}$ for all $f \in \mathcal{S}$ and $y \in G$ where $L_y f(x) = f(y^{-1}x)$.
- (4) For all $f \in \mathcal{S}$, the mapping $y \longrightarrow L_y f$ is continuous.

In [1], J. T. Burnham with changing $L^1(G)$ by an arbitrary Banach algebra A , gave a generalization of Segal algebras and introduced the notion of an abstract Segal algebra.

It is well-known that $L^1(G)$ is a commutative semi-simple regular Banach algebra with a bounded approximate identity with compact support; see [10].

Recently, J. Inoue and S.-E. Takahasi in [6] investigated abstract Segal algebras in a non-unital commutative semi-simple regular Banach algebra A such that A has a bounded approximate identity in A_c where

$$A_c = \{a \in A : \hat{a} \text{ has compact support}\},$$

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and \widehat{a} denotes the Gel'fand transform of a . Indeed, they gave the following definition of a Segal algebra in A .

Definition 1.1. An ideal \mathcal{S} in A is called a Segal algebra in A if it satisfies the following properties:

- (1) \mathcal{S} is dense in A .
- (2) \mathcal{S} is a Banach space under some norm $\|\cdot\|_{\mathcal{S}}$ such that $\|a\|_A \leq \|a\|_{\mathcal{S}}$ for each $a \in \mathcal{S}$.
- (3) $\|ax\|_{\mathcal{S}} \leq \|a\|_A \|x\|_{\mathcal{S}}$ for each $a \in A$ and $x \in \mathcal{S}$.
- (4) \mathcal{S} has approximate units.

Clearly, an abstract Segal algebra in A (in the sense of Burnham) is a Segal algebra in A if and only if it possesses approximate units.

A commutative Banach algebra A is without order if for $a \in A$, the condition $aA = \{0\}$ implies $a = 0$ or equivalently A does not have any non-zero annihilator. For example if A has an approximate identity, then it is without order. A linear operator T on A is called a multiplier if it satisfies $aT(b) = T(a)b$ for all $a, b \in A$. Suppose that $\mathcal{M}(A)$ denotes the space of all multipliers of the Banach algebra A . If $\Delta(A)$ denotes the space of all characters of A ; that is, non-zero homomorphisms from A into \mathbb{C} , then for each $T \in \mathcal{M}(A)$, there exists a unique bounded continuous function \widehat{T} on $\Delta(A)$ such that $\widehat{T(a)}(\phi) = \widehat{T}(\phi)\widehat{a}(\phi)$ for all $a \in A$ and $\phi \in \Delta(A)$; see [10, Proposition 2.2.16]. Let $\widehat{\mathcal{M}(A)}$ denote the space of all \widehat{T} corresponding to $T \in \mathcal{M}(A)$.

A bounded continuous function σ on $\Delta(A)$ is called a BSE-function if there exists a constant $C > 0$ such that for each $\phi_1, \dots, \phi_n \in \Delta(A)$ and complex numbers c_1, \dots, c_n , the inequality

$$\left| \sum_{i=1}^n c_i \sigma(\phi_i) \right| \leq C \left\| \sum_{i=1}^n c_i \phi_i \right\|_{A^*}$$

holds. Let $C_{\text{BSE}}(\Delta(A))$ be the set of all BSE-functions.

A without order commutative Banach algebra A is called a BSE-algebra if

$$C_{\text{BSE}}(\Delta(A)) = \widehat{\mathcal{M}(A)}.$$

The theory of BSE-algebras for the first time introduced and investigated by Takahasi and Hatori; see [15] and two other notable works [11, 9]. In [9], the authors answered to a question raised in [15]. Examples of BSE-algebras are the group algebra $L^1(G)$ of a locally compact abelian group G , the Fourier algebra $A(G)$ of a locally compact amenable group G , all commutative C^* -algebras, the disk algebra, and the Hardy algebra on the open unit disk.

A net $\{a_\alpha\}$ in A is called a bounded weak approximate identity (b.w.a.i) for A if $\{a_\alpha\}$ is bounded in A and

$$\lim_{\alpha} \phi(a_\alpha a) = \phi(a) \quad (\phi \in \Delta(A), a \in A),$$

or equivalently, $\lim_{\alpha} \phi(a_\alpha) = 1$ for each $\phi \in \Delta(A)$. Clearly, each b.a.i of A is a b.w.a.i and the converse is not valid in general; see [7] and [12]. Note that bounded weak approximate identities are important to decide whether a commutative Banach algebra is a BSE-algebra or not. For example, it was shown in [5] that a Segal algebra $S(G)$ on a locally compact abelian group G is a BSE-algebra if and only if it has a bounded weak approximate identity.

For undefined concepts and notations appearing in the sequel, one can consult [2, 10].

The outline of the next sections is as follows:

In §2, for a commutative semi-simple Banach algebra A which is an ideal in its second dual we give a necessary and sufficient conditions for an essential abstract Segal algebra in A to be a BSE-algebra. We show that a large class of abstract Segal algebras in the Fourier algebra $A(G)$ of a locally compact group G are BSE-algebra if and only if they have bounded weak approximate identities. In §3, we study the BSE-property of the Segal algebra $A_{\tau(n)}$ in A which introduced by Inoue and Takahasi and in the case that $(A, \|\cdot\|_X)$ is a BSE-algebra, we show that $A_{\tau(n)}$ is a BSE-algebra if and only if τ is bounded where $\tau : X \rightarrow \mathbb{C}$ is a certain continuous function. Also, we compare the BSE-property between A and $A_{\tau(n)}$. In §4, motivated by the definition of $A_{\tau(n)}$, for an arbitrary locally compact (abelian) group G , and a measurable sub-multiplicative function $\tau : G \rightarrow \mathbb{C}^\times$, we define the Banach algebra $L^1(G)_{\tau(n)}$. Then we investigate the BSE-property of this algebra.

2. BSE-ABSTRACT SEGAL ALGEBRAS

Recall that a Banach algebra B is an abstract Segal algebra of a Banach algebra A if

- (1) B is a dense left ideal in A ,
- (2) there exists $M > 0$ such that $\|b\|_A \leq M\|b\|_B$ for each $b \in B$
- (3) there exists $C > 0$ such that $\|ab\|_B \leq C\|a\|_A\|b\|_B$ for each $a, b \in B$.

Endow $\Delta(A)$ and $\Delta(B)$ with the Gel'fand topology, the map $\varphi \mapsto \varphi|_B$ is a homeomorphism from $\Delta(A)$ onto $\Delta(B)$; see [1, Theorem 2.1].

Let B be an abstract Segal algebra with respect to A . We say that B is essential if $\langle AB \rangle$ is $\|\cdot\|_B$ -dense in B , where $\langle AB \rangle$ is the linear span of $AB = \{ab : a \in A, b \in B\}$.

Theorem 2.1. *Let A be a semi-simple commutative Banach algebra which is an ideal in its second dual A^{**} . Suppose that B is an essential abstract Segal algebra in A . Then the following statements are equivalent.*

- (i) B is a BSE-algebra.
- (ii) $B = A$ and A is a BSE-algebra.

Proof. Suppose that B is a BSE-algebra. Then by [15, Corollary 5], B has a b.w.a.i, say $(g_\gamma)_\gamma$. It is clear that $(g_\gamma)_\gamma$ is also a b.w.a.i for A . So, by [11, Theorem 3.1] A is a BSE-algebra and has a bounded approximate identity, say $(e_\alpha)_\alpha$. Since B is essential, $(e_\alpha)_\alpha$ is also a bounded approximate identity for B in A . In fact, for each $b \in B$ and $\varepsilon > 0$, there is $c = \sum_{i=1}^n a_i b_i$ with $1 \leq i \leq n$, $a_i \in A$ and $b_i \in B$ such that $\|b - c\|_B \leq \varepsilon$. Thus for each α we have

$$\|e_\alpha b - b\|_B \leq (1 + K)\varepsilon + C \sum_{i=1}^n \|e_\alpha a_i - a_i\|_A \|b_i\|_B,$$

where $K = \sup \|e_\alpha\|_A$. This shows that $\|e_\alpha b - b\|_B \rightarrow 0$ for all $b \in B$. Thus, $B = BA$ by Cohen's factorization theorem. Now, let $b \in B$ and (b_n) be a bounded sequence in B . Then $b = ca$ for some $c \in B$ and $a \in A$. Since the sequence (b_n) is also bounded in A and A is an ideal in its second dual, it follows that the operator $\rho_a : A \rightarrow A$ defined by $\rho_a(a') = aa'$, ($a' \in A$) is weakly compact; see [3]. Therefore, there exists a subsequence (b_{n_k}) of (b_n) such that $(\rho_a(b_{n_k}))$ is convergent to some a' in the weak topology of A . Now, we observe that $f \cdot c \in A^*$ for all $f \in B^*$, where $(f \cdot c)(a) = f(ca)$ for all $a \in A$. This shows that the sequence $(\rho_{ca}(b_{n_k}))$ is convergent to ca' in the weak topology of B . Therefore, the operator $\rho_b : B \rightarrow B$ is weakly compact which implies that B is an ideal in its second dual. Since B is semi-simple, [11, Theorem 3.1] implies that B has a bounded approximate identity. Thus $A = B$ by [1, Theorem 1.2]. That (ii) implies (i) is trivial. \square

Example 2.2. Let G be a locally compact group and let $A(G)$ be the Fourier algebra of G . It was shown in [11, Theorem 5.1] that $A(G)$ is a BSE-algebra if and only if G is amenable. Moreover, $A(G)$ is an ideal in its second dual if and only if G is discrete; see [4, Lemma 3.3]. Therefore, by Theorem 2.1 if G is discrete, then each essential abstract Segal algebra $SA(G)$ in $A(G)$ is a BSE-algebra if and only if $SA(G) = A(G)$ and G is amenable.

Let G be a locally compact group and let $L^r(G)$ be the Lebesgue L^r -space of G , where $1 \leq r < \infty$. Then

$$SA^r(G) := L^r(G) \cap A(G)$$

with the norm $|||f||| = \|f\|_r + \|f\|_{A(G)}$ and the pointwise product is an abstract Segal algebra in $A(G)$.

Corollary 2.3. *Let G be a discrete group and let $1 \leq r \leq 2$. Then $SA^r(G)$ is a BSE-algebra if and only if G is finite*

Proof. If G is finite, then $SA^r(G) = A(G)$. So, the result follows from Theorem 2.1.

For the converse, first note that $SA^r(G) = l^r(G)$ and the norms $\|\cdot\|_r$ and $|||\cdot|||$ on $SA^r(G)$ are equivalent by the open mapping theorem. In fact, $l^2(G) \subseteq \delta_e * l^2(G) \subseteq A(G)$, where δ_e is the point mass at the identity element e of G . So, if $1 \leq r \leq 2$, then $l^r(G) \subseteq l^2(G)$ and

$$l^r(G) = l^r(G) \cap l^2(G) \subseteq SA^r(G) \subseteq l^r(G).$$

Moreover, it is clear that $l^r(G)$ has an approximate identity and consequently it is an essential abstract Segal algebra in $A(G)$. Therefore, if $SA^r(G) = l^r(G)$ is a BSE-algebra, then $A(G) = l^r(G)$ by Example 2.2. Thus, $A(G) = l^2(G)$, is a reflexive predual of a W^* -algebra. This implies, as is known, that $A(G)$ is finite dimensional; see [13]. Thus G is finite, which completes the proof. \square

For a locally compact group G , we recall that $A(G)$ is always an ideal in the Fourier-Stieltjes algebra $B(G)$ and note that $\mathcal{M}(A(G)) = B(G)$ when G is amenable. The spectrum of $A(G)$ can be canonically identified with G . More precisely, the map $x \rightarrow \varphi_x$ where $\varphi_x(u) = u(x)$ for all $u \in A(G)$ is a homeomorphism from G onto $\Delta(A(G))$.

Theorem 2.4. *Let G be a locally compact group and let $SA(G)$ be an abstract Segal algebra in $A(G)$ such that $B(G) \subseteq \mathcal{M}(SA(G))$. Then $SA(G)$ is a BSE-algebra if and only if $SA(G)$ has a b.w.a.i.*

Proof. Clearly if $SA(G)$ is a BSE-algebra, then it has a b.w.a.i.

Conversely, suppose that $SA(G)$ has a b.w.a.i, say $(e_\gamma)_\gamma$. Then $\widehat{\mathcal{M}(SA(G))} \subseteq C_{BSE}(\Delta(SA(G)))$ by [15, Corollary 5]. Moreover, it is clear that $(e_\gamma)_\gamma$ is also a b.w.a.i for $A(G)$. Consequently, we conclude that G is amenable by [11, Theorem 5.1]. Now, we need to show the reverse inclusion. Since $SA(G)$ is an abstract Segal algebra in $A(G)$, there exists $M > 0$ such that $\|u\|_{A(G)} \leq M\|u\|_{SA(G)}$ for all $u \in SA(G)$. Thus, for any $x_1, \dots, x_n \in G$ and $c_1, \dots, c_n \in \mathbb{C}$,

$$\left\| \sum_{j=1}^n c_j \varphi_{x_j} \right\|_{SA(G)^*} \leq M \left\| \sum_{j=1}^n c_j \varphi_{x_j} \right\|_{A(G)^*}.$$

This implies that

$$\begin{aligned} C_{BSE}(\Delta(SA(G))) &\subseteq C_{BSE}(\Delta(A(G))) \\ &= \widehat{B(G)} \\ &\subseteq \widehat{\mathcal{M}(SA(G))}. \end{aligned}$$

Hence, $SA(G)$ is a BSE-algebra. \square

Example 2.5. (1) Let G be a locally compact group and let $1 \leq r < \infty$. Now, since $\|u\|_\infty \leq \|u\|_{B(G)}$ for all $u \in B(G)$, it follows that $uL^r(G) \subseteq L^r(G)$. This implies that $B(G) \subseteq \mathcal{M}(SA^r(G))$. Thus $SA^r(G)$ is a BSE-algebra if and only if it has a b.w.a.i.

(2) Let $S_0(G)$ be the Feichtinger Segal algebra in $A(G)$. Then $B(G) \subseteq \mathcal{M}(S_0(G))$; see [14, Corollary 5.2]. Thus $S_0(G)$ is a BSE-algebra if and only if it has a b.w.a.i.

Corollary 2.6. *Let G be a locally compact group and let $SA(G)$ be an essential abstract Segal algebra in $A(G)$. Then $SA(G)$ is a BSE-algebra if and only if $SA(G)$ has a b.w.a.i.*

Proof. Suppose that $SA(G)$ has a b.w.a.i. Then $A(G)$ has a bounded approximate identity. Now, by a same argument during the proof of Theorem 2.1 we can show that $SA(G) = A(G)SA(G)$. Consequently,

$$uSA(G) = uA(G)SA(G) \subseteq A(G)SA(G) = SA(G)$$

for all $u \in B(G)$ which implies that $B(G) \subseteq \mathcal{M}(SA(G))$. Hence, $SA(G)$ is a BSE-algebra by Theorem 2.4. \square

3. SEGAL ALGEBRAS IMPLEMENTED BY LOCAL FUNCTIONS

In this section we focus on a certain Segal algebra which recently introduced by Inoue and Takahasi. Let X be a non-empty locally compact Hausdorff space. A subalgebra A of $C_0(X)$ is called a Banach function algebra if A separates strongly the points of X (that is, for each $x, y \in X$ with $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$) and for each $x \in X$, there exists $f \in A$ such that $f(x) \neq 0$ and with a norm $\|\cdot\|$, $(A, \|\cdot\|)$ is a Banach algebra.

Suppose that $(A, \|\cdot\|)$ is a natural regular Banach function algebra on a locally compact, non-compact Hausdorff space X with a bounded approximate identity $\{e_\alpha\}$ in A_c . We recalling the following definitions from [6].

Definition 3.1. A complex-valued continuous function σ on X is called a local A -function if for all $f \in A_c$, $f\sigma \in A$. The set of all local A -functions is denoted by A_{loc}

Definition 3.2. For positive integer n and a continuous complex-valued function τ on A , put

$$A_{\tau(n)} = \{f \in A : f\tau^k \in A \quad (0 \leq k \leq n)\},$$

$$\|f\|_{\tau(n)} = \sum_{k=0}^n \|f\tau^k\|.$$

In the sequel of this section, suppose that n is a constant positive integer and $\tau \in A_{\text{loc}}$.

By [6, Theorems 5.4], if $\tau \in A_{\text{loc}}$, then $(A_{\tau(n)}, \|\cdot\|_{\tau(n)})$ is a Segal algebra in A such that $\Delta(A_{\tau(n)}) = \Delta(A) = X$, that is, $x \longrightarrow \phi_x$ is a homeomorphism from X onto $\Delta(A_{\tau(n)})$.

Also, one can see that $A_{\tau(n)}$ is a Banach function algebra, because for each $x, y \in X$ with $x \neq y$, there exists $f \in A$ such that $f(x) = \phi_x(f) \neq \phi_y(f) = f(y)$ and by using the Urysohn lemma for each $x \in X$, there exists $f \in A$ with $f(x) \neq 0$. Note that by [6, Theorem 3.5], $A_{\tau(n)}$ is Tauberian. Recall that a Banach algebra A is Tauberian if A_c is dense in A .

The following theorem is one of our main result in this section.

Theorem 3.3. *Suppose that $(A, \|\cdot\|_X)$ is a BSE-algebra where $\|\cdot\|_X$ is the uniform norm. Then the following statements are equivalent.*

- (i) $A_{\tau(n)}$ is a BSE-algebra.
- (ii) τ is bounded.

Proof. (i) \rightarrow (ii). Suppose that $A_{\tau(n)}$ is a BSE-algebra, therefore it has a b.w.a.i. So, there exists a constant $M > 0$ such that

$$\|f_\alpha\|_{\tau(n)} < M, \quad \lim_{\alpha} f_\alpha(x) = 1 \quad (x \in X).$$

On the other hand, $\|f_\alpha\tau\|_X \leq \|f_\alpha\|_{\tau(n)}$, hence we have

$$|\tau(x)| \leq M \quad (x \in X).$$

Therefore, τ is bounded.

(ii) \rightarrow (i). Let τ be bounded by M , that is, $\|\tau\|_X < M$. Clearly, $A_c \subseteq A_{\tau(n)}$. For each $f \in A$, there exists a net $\{f_\alpha\}$ in A_c such that $\|f_\alpha - f\|_X \longrightarrow 0$. Now, $\{f_\alpha\tau\}$ is in A and $f\tau = \lim_{\alpha} f_\alpha\tau$, since

$$\|f\tau - f_\alpha\tau\|_X = \|(f - f_\alpha)\tau\|_X \leq M\|f - f_\alpha\|_X \longrightarrow 0.$$

So, $f\tau \in A$. Similarly, one can see that $f\tau^k$ for each $1 < k \leq n$ is in A . Therefore, $f \in A_{\tau(n)}$. Hence $A = A_{\tau(n)}$. Finally, for each $f \in A$

$$\|f\| \leq \|f\|_{\tau(n)} \leq \|f\| \left(\sum_{k=0}^n M^k \right).$$

Therefore, $A_{\tau(n)}$ is topologically isomorphic to A , so $A_{\tau(n)}$ is a BSE-algebra. \square

Recall that $\tau \in A_{\text{loc}}$ is called a rank ∞ local A -function, if for each $k = 0, 1, 2, \dots$, the inclusion $A_{\tau(k)} \supsetneq A_{\tau(k+1)}$ holds. By [6, Proposition 8.2 (ii)], if $\|\tau\|_X = \infty$, then τ is a rank ∞ local A -function.

As an application of the above theorem, we give the following result which provide for us examples of Banach algebras without any b.w.a.i.

Corollary 3.4. *Let $A = C_0(\mathbb{R})$ and $\tau(x) = x$ for every $x \in \mathbb{R}$. Then*

$$A \supsetneq A_{\tau(1)} \supsetneq A_{\tau(2)} \supsetneq \dots \supsetneq A_{\tau(n)} \supsetneq \dots \quad (3.1)$$

For each $k = 1, 2, 3, \dots$, $A_{\tau(k)}$ is not a BSE-algebra and has no b.w.a.i. .

Proof. By [15, Theorem 3], if $A = C_0(\mathbb{R})$, we know that A is a BSE-algebra. Since for each $x \in \mathbb{R}$, there exists $f \in A$ such that $f = \tau$ on a neighborhood of x , hence by [6, Proposition 7.2], τ is an element of A_{loc} . But τ is not bounded and this implies that $A_{\tau(k)}$ is not a BSE-algebra by Theorem 3.3. Also, since τ is not bounded, τ is a rank ∞ local A -function, hence we have equation 3.1. Finally, if $A_{\tau(k)}$ has a b.w.a.i., then similar to the proof of Theorem 3.3 one has $\|\tau\|_{\mathbb{R}} < \infty$ which is impossible. \square

By the above corollary if A is a BSE-algebra, then $A_{\tau(n)}$ is not necessarily a BSE-algebra. For the converse we have the following proposition.

Proposition 3.5. *Suppose that $\tau \in A_{\text{loc}}$ and A is an ideal in its second dual. If $A_{\tau(n)}$ is a BSE-algebra, then A is a BSE-algebra.*

Proof. If $\{e_\alpha\}$ is a b.a.i for A , then $\{e_\alpha\}$ is an approximate identity for $A_{\tau(n)}$. So, $A_{\tau(n)}$ is an essential abstract algebra with respect to A . Now, using Theorem 2.1 we have the result. \square

We do not know whether Proposition 3.5 fails if the assumption that A is an ideal in its second dual is dropped.

Remark 3.6. If X is a discrete space, then A is an ideal in its second dual. Since A is a semi-simple commutative and Tauberian Banach algebra, using [11, Remark 3.5], we conclude that A is an ideal in its second dual.

4. A CONSTRUCTION ON GROUP ALGEBRAS

Let G be a locally compact group and let $L^1(G)$ be the space of all measurable and integrable complex-valued functions (equivalent classes with respect to the

almost everywhere equality relation) on G with respect to the left Haar measure of G . The convolution product of the functions f and g in $L^1(G)$ is defined by

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy.$$

For each $f \in L^1(G)$, let $\|f\|_1 = \int_G |f(x)|dx$. It is well-known that $L^1(G)$ endowed with the norm $\|\cdot\|_1$ and the convolution product is a Banach algebra called the group algebra of G ; see [2, Section 3.3] for more details.

The following lemma proved by Bochner and Schoenberg for $G = \mathbb{R}$ in (1934) and by Eberlein for general locally compact abelian (LCA) groups in (1955). Here we give another proof with using a result due to E. Kaniuth and A. Ülger. Recall that for a locally compact group G , $A(G)$ denotes the Fourier algebra defined by P. Eymard in (1964); see [10, section 2.9] for more details.

Recall that the dual group of G , \widehat{G} defined as the set of all continuous homomorphisms from G to \mathbb{T} where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. It is well-known that \widehat{G} is a LCA group with pointwise operation.

Lemma 4.1. *Suppose that G is LCA group. Then $L^1(G)$ is a BSE-algebra.*

Proof. Let G be a LCA group with dual space \widehat{G} . If H is a locally compact group, then by [11, Theorem 5.1], $A(H)$ is a BSE-algebra if and only if H is amenable. On the other hand, we know that $L^1(G)$ is isometrically isomorphic to $A(\widehat{G})$. But \widehat{G} is amenable. Therefore, $A(\widehat{G})$ and so $L^1(G)$ is a BSE-algebra. \square

In the sequel, motivated by the construction of $A_{\tau(n)}$ in the preceding section, we introduce a subalgebra of the group algebra $A = L^1(G)$ where G is a locally compact group.

Recall that $\varphi : G \longrightarrow \mathbb{C}^\times$ is sub-multiplicative if

$$|\varphi(xy)| \leq |\varphi(x)||\varphi(y)| \quad (x, y \in G),$$

where \mathbb{C}^\times denotes the multiplicative group of non-zero complex numbers.

For a measurable sub-multiplicative function $\tau : G \longrightarrow \mathbb{C}^\times$ and each $n \in \mathbb{N}$, put

$$\begin{aligned} L^1(G)_{\tau(n)} &= \{f \in L^1(G) : f\tau, \dots, f\tau^n \in L^1(G)\} \\ \|f\|_{\tau(n)} &= \sum_{k=0}^n \|f\tau^k\|_1 \quad (f \in L^1(G)_{\tau(n)}). \end{aligned}$$

Proposition 4.2. *$L^1(G)_{\tau(n)}$ is a Banach algebra with the convolution product and the norm $\|\cdot\|_{\tau(n)}$.*

Proof. For each $f, g \in L^1(G)_{\tau(n)}$ and $1 \leq k \leq n$, we have

$$\begin{aligned} \|(f * g)\tau^k\|_1 &\leq \int \int |f(y)| |g(y^{-1}x)| |\tau^k(x)| dy dx \\ &= \int \int |f(y)| |g(y^{-1}x)| |\tau^k(x)| dx dy \\ &= \int \int |f(y)| |g(x)| |\tau^k(yx)| dx dy \\ &\leq \|f\tau^k\|_1 \|g\tau^k\|_1. \end{aligned}$$

For each $1 \leq k \leq n$, since τ is measurable, $(f * g)\tau^k$ is measurable and by the above inequality $\|(f * g)\tau^k\|_1 < \infty$. Therefore, $f * g$ is in $L^1(G)_{\tau(n)}$. Also, we have

$$\begin{aligned} \|f * g\|_{\tau(n)} &= \|f * g\|_1 + \sum_{k=1}^n \|(f * g)\tau^k\|_1 \\ &\leq \|f\|_1 \|g\|_1 + \sum_{k=1}^n \|f\tau^k\|_1 \|g\tau^k\|_1 \\ &\leq \|f\|_{\tau(n)} \|g\|_{\tau(n)}. \end{aligned}$$

To see the completeness of $\|\cdot\|_{\tau(n)}$, let $\{f_i\}$ be a Cauchy sequence in $L^1(G)_{\tau(n)}$. So, there exist $f \in L^1(G)$ and $g_k \in L^1(G)$ for each $1 \leq k \leq n$ such that

$$\lim_{i \rightarrow \infty} \|f_i - f\|_1 = 0, \quad \lim_{i \rightarrow \infty} \|f_i \tau^k - g_k\|_1 = 0.$$

Since $\lim_{i \rightarrow \infty} \|f_i - f\|_1 = 0$, there exists a subsequence $\{f_{i_m}\}$ such that

$$\lim_{i_m} f_{i_m}(x) = f(x) \text{ a.e., for } x \in G.$$

Also, since $\lim_{i_m} \|f_{i_m} \tau^k - g_k\|_1 = 0$, there exists a subsequence $\{f_{i_{m,k}}\}$ of $\{f_{i_m}\}$ such that

$$\lim_{i_{m,k}} f_{i_{m,k}}(x) \tau^k(x) = g_k(x) \text{ a.e., for } x \in G.$$

Therefore, $f\tau^k = g_k$ a.e., so, f is an element of $L^1(G)_{\tau(n)}$ such that

$$\begin{aligned} \|f_i - f\|_{\tau(n)} &= \|f_i - f\|_1 + \|f_i \tau - f\tau\|_1 + \cdots + \|f_i \tau^n - f\tau^n\|_1 \\ &= \|f_i - f\|_1 + \|f_i \tau - g_1\|_1 + \cdots + \|f_i \tau^n - g_n\|_1 \longrightarrow 0. \end{aligned}$$

Hence, $(L^1(G)_{\tau(n)}, \|\cdot\|_{\tau(n)})$ is complete. \square

In the sequel G is a LCA group.

Theorem 4.3. *If τ is bounded, then $L^1(G)_{\tau(n)}$ is a BSE-algebra.*

Proof. If τ is bounded by M , then for each $f \in L^1(G)$ and $1 \leq k \leq n$ we have $f\tau^k \in L^1(G)$ and

$$\|f\|_1 \leq \|f\|_{\tau(n)} = \sum_{k=0}^n \|f\tau^k\|_1 \leq \|f\|_1 \left(\sum_{k=0}^n M^k \right).$$

So, $L^1(G)_{\tau(n)}$ is topologically isomorphic to $L^1(G)$ and hence it is a BSE-algebra by Lemma 4.1. \square

When τ satisfying $|\tau(x)| \geq 1$ a.e., for $x \in G$, we show in the sequel that $L^1(G)_{\tau(n)}$ is indeed a Beurling algebra which we recall its definition as follows:

A weight on G is a measurable function $w : G \rightarrow (0, \infty)$ such that $w(xy) \leq w(x)w(y)$ for all $x, y \in G$. The Beurling algebra $L^1(G, w)$ defined by the space of all measurable and complex-valued functions f on G such that $\|f\|_{1,w} = \int |f(x)|w(x) dx < \infty$. The Beurling algebra with the convolution product and the norm $\|\cdot\|_{1,w}$ is a Banach algebra with $\Delta(L^1(G, w)) = \widehat{G}(w)$, where $\widehat{G}(w)$ is the space of all non-zero complex-valued continuous homomorphisms φ on G such that $|\varphi(x)| \leq w(x)$ for each $x \in G$; see [10].

The space $M(G, w)$ of all complex regular Borel measures μ on G such that $\mu w \in M(G)$ with convolution product and norm $\|\mu\|_{M(G)} = \int w(x) d|\mu|(x)$ is a Banach algebra called the weighted measure algebra, where μw defined by

$$\mu w(B) = \int_B w(x) d\mu(x) \quad \text{for each Borel subset } B \text{ of } G.$$

Proposition 4.4. *If $|\tau| \geq 1$ a.e., then $L^1(G)_{\tau(n)}$ and $L^1(G, |\tau^n|)$ are topologically isomorphic.*

Proof. Suppose that for almost every $x \in G$, $|\tau(x)| \geq 1$. Clearly if $f \in L^1(G)_{\tau(n)}$, then we have $f \in L^1(G, |\tau^n|)$. On the other hand, if $f \in L^1(G, |\tau^n|)$ by applying $|\tau| \geq 1$ a.e., we conclude that $f \in L^1(G)_{\tau(n)}$, since for all $0 \leq k \leq n$, $|f(x)\tau^k(x)| \leq |f(x)|\tau^n(x)$ a.e., therefore, $L^1(G, |\tau^n|) = L^1(G)_{\tau(n)}$ as two sets. Also, using $|\tau| \geq 1$ a.e., we have

$$\|f\|_{1,w} \leq \|f\|_{\tau(n)} \leq n\|f\|_{1,w},$$

where $w = |\tau^n|$. So, two norms $\|\cdot\|_{1,w}$ and $\|\cdot\|_{\tau(n)}$ are equivalent, which completes the proof. \square

Remark 4.5. Note that $L^1(G)_{\tau(n)}$ and $L^1(G, |\tau^n|)$ are not equal in general. For example, let $G = R^+$ be the multiplicative group of all positive real numbers, $n = 2$ and $\tau(x) = \frac{1}{x}$ for all $x \in R^+$. Clearly, τ is measurable and sub-multiplicative. Also, it is easily verified that $L^1(G)_{\tau(2)} \subseteq L^1(G, |\tau^2|)$. Now, take $0 < \alpha < 1$ and

put

$$f(x) = \begin{cases} 0, & 0 < x < 1; \\ x^\alpha, & 1 \leq x. \end{cases}$$

One can easily check that f is in $L^1(G, |\tau^2|)$ but it is not in $L^1(G)_{\tau(2)}$. So,

$$L^1(G, |\tau^2|) \neq L^1(G)_{\tau(2)}.$$

Also, if we put $g(x) = \chi_{(0,1]}$, then $g \in L^1(G)$ but $g \notin L^1(G)_{\tau(2)}$. Hence,

$$L^1(G) \neq L^1(G)_{\tau(2)}.$$

Remark 4.6. Although in general $L^1(G, |\tau^k|) \neq L^1(G)_{\tau(n)}$ for every integer k with $0 \leq k \leq n$, but we have

$$\overline{L^1(G)_{\tau(n)}}^{\|\cdot\|_{1,|\tau^k|}} = L^1(G, |\tau^k|), \quad (0 \leq k \leq n).$$

Because $C_c(G)$ is dense in $L^1(G, |\tau^k|)$ and similar to [10, Lemma 1.3.5 (i)], one can see that $C_c(G) \subseteq L^1(G)_{\tau(n)}$.

Remark 4.7. Let $K \subseteq G$ be a relatively compact neighborhood of e ; the identity of G . Put

$$\mathcal{U}_K = \{U \subseteq K : U \text{ is a relatively compact neighborhood of } e\}.$$

For each $U \in \mathcal{U}_K$, let $f_U = \frac{\chi_U}{|U|}$, where $|U|$ denotes the Haar measure of U . On the other hand, since K is relatively compact by [10, Lemma 1.3.3], there exists a positive real number b such that $|\tau(x)| \leq b$ for all $x \in K$. So

$$\|f_U\|_{\tau(n)} \leq 1 + b + \dots + b^n \quad (U \in \mathcal{U}_K).$$

Also, similar to the group algebra case, for each $f \in L^1(G)_{\tau(n)}$ we have $\|f_U * f - f\|_{\tau(n)} \rightarrow 0$ when U tends to $\{e\}$. Therefore, $\{f_U\}_{U \in \mathcal{U}_K}$ is a b.a.i for $L^1(G)_{\tau(n)}$.

Remark 4.8. Using the above remarks one can see that in general $L^1(G)_{\tau(n)}$ is not an abstract Segal algebra with respect to $L^1(G)$, because $L^1(G)_{\tau(n)}$ has a b.a.i and in general $L^1(G) \neq L^1(G)_{\tau(n)}$. But it is well-known that if \mathcal{S} is an abstract Segal algebra with respect to $L^1(G)$ such that has a b.a.i, then $\mathcal{S} = L^1(G)$.

Suppose that G is compact and w is a weight on G . So, by Lemma 1.3.3 and Corollary 1.3.4 of [10], there exists positive real number b such that $1 \leq w(x) \leq b$ for all $x \in G$. Hence, $L^1(G, w)$ is topologically isomorphic to $L^1(G)$. Now, using Proposition 4.4 and the proof of Theorem 4.3, we have the following corollary. Note that " \cong " means topologically isomorphic.

Corollary 4.9. *If G is a compact group, then $L^1(G)_{\tau(n)}$ is a BSE-algebra and we have the following relations:*

$$L^1(G, |\tau^n|) \cong L^1(G)_{\tau(n)} \cong L^1(G).$$

Remark 4.10. For every integer k with $0 \leq k \leq n$, we have,

$$L^1(G)_{\tau(n)} \subseteq L^1(G, |\tau^k|).$$

Therefore,

$$\widehat{G} \cup \widehat{G}(|\tau|) \cup \dots \cup \widehat{G}(|\tau^n|) \subseteq \Delta(L^1(G)_{\tau(n)}).$$

It would be interesting to give a complete characterization of $\Delta(L^1(G)_{\tau(n)})$. Note that by the above relation, we conclude that $L^1(G)_{\tau(n)}$ is semi-simple.

Remark 4.11. Put $M(G)_{\tau(n)} = \bigcap_{k=0}^n M(G, |\tau^k|)$ and define the following norm:

$$\|\mu\|_{\tau(n)} = \sum_{k=0}^n \|\mu \cdot |\tau^k|\|_{M(G)} \quad (\mu \in M(G)_{\tau(n)}).$$

A direct use of the convolution product shows that $(M(G)_{\tau(n)}, \|\cdot\|_{\tau(n)})$ is a normed algebra such that

$$M(G)_{\tau(n)} \subseteq \mathcal{M}(L^1(G)_{\tau(n)}).$$

We do not know whether the converse of the above inequality is hold or not. Clearly, if τ is bounded, then $M(G) \cong M(G)_{\tau(n)}$ and so the converse of the above inequality holds by Wendel's Theorem.

As it is shown in [8, Remark 3.2], if B is an abstract Segal algebra with respect to A , then $C_{\text{BSE}}(\Delta(B)) \subseteq C_{\text{BSE}}(\Delta(A))$. If $A = L^1(G)$ and $B = L^1(G)_{\tau(n)}$ although B is not an abstract Segal algebra with respect to A in general, but we have a similar result as follows. We prove it for the sake of completeness and convenience of the reader.

Proposition 4.12. *Let $A = L^1(G)$ and $B = L^1(G)_{\tau(n)}$. Then we have*

$$\widehat{\mathcal{M}(B)} \subseteq C_{\text{BSE}}(\Delta(B)) \subseteq C_{\text{BSE}}(\Delta(A)) = \widehat{\mathcal{M}(A)}.$$

Proof. Since B has a b.a.i, $\widehat{\mathcal{M}(B)} \subseteq C_{\text{BSE}}(\Delta(B))$.

To see the second inequality, let $f \in A^*$. In view of the following relation

$$|f(b)| \leq \|f\|_{A^*} \|b\|_A \leq \|f\|_{A^*} \|b\|_{\tau(n)} \quad (b \in B),$$

we have

$$\|f\|_{B^*} \leq \|f\|_{A^*}. \quad (4.1)$$

Now, let $\sigma \in C_{\text{BSE}}(\Delta(B))$, so there exists $C > 0$ such that for each $\varphi_1, \dots, \varphi_n \in \Delta(B)$ and complex numbers c_1, \dots, c_n

$$\left| \sum_{i=1}^n c_i \sigma(\varphi_i) \right| \leq C \left\| \sum_{i=1}^n c_i \varphi_i \right\|_{B^*}.$$

If for each $1 \leq i \leq n$ we take $\varphi_i \in \Delta(A) = \widehat{G} \subseteq \Delta(B)$ and using relation 4.1 we conclude that

$$\left| \sum_{i=1}^n c_i \sigma(\varphi_i) \right| \leq C \left\| \sum_{i=1}^n c_i \varphi_i \right\|_{B^*} \leq C \left\| \sum_{i=1}^n c_i \varphi_i \right\|_{A^*}.$$

Hence σ is an element of $C_{\text{BSE}}(\Delta(A))$, which completes the proof. \square

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Mohammad Fozouni

Department of Mathematics, Gonbad Kavous University, P.O.Box 163,
Gonbad-e Kavous, Iran.
Email: fozouni@gonbad.ac.ir

Mehdi Nemati

Department of Mathematical Sciences, Isfahan University of Technology
Isfahan 84156-83111, Iran

Email: m.nemati@cc.iut.ac.ir